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VARIATIONAL METHODS OF CONSTRUCTING CHAOTIC MOTIONS IN RIGID-BODY DYNAMICS†

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A method based on the variational calculus "in the large" (Morse theory) and [1], is proposed for proving the existence of chaotic motions in Hamiltonian systems with two degrees of freedom. The method is used to analyse the motion of a dynamically symmetric heavy rigid body with its centre of mass in a diametral plane. Dynamically asymmetric bodies may be treated similarly.

The treatment of non-integrable systems by perturbation methods is feasible only in near-integrable systems [2]. In rigid body dynamics such methods may be used to establish non-integrability and chaotic behaviour of solutions only for parameter values not far from those corresponding to known integrable cases. The method proposed below is free from such restrictions.

1. CHAOTIC MOTIONS OF A RIGID BODY

THE EULER–POISSON equations for the motion of a rigid body are

$$J\omega^{\cdot}=[J\omega,\ \omega]+[\gamma,\ e]\quad \gamma^{\cdot}=[\gamma,\ \omega] \tag{1.1}$$

where γ is the Poisson unit vector, ω is the angular velocity vector, J is the inertia tensor and e is the product of the weight of the body and the radius-vector of its centre of mass. Equations (1.1) are Hamiltonian on the four-dimensional level set

$$M=\{(J\omega,\ \gamma)=c,\ \ |\gamma|=1\}\subseteq R^6 \tag{1.2}$$

of the area integral and have an energy integral

$$H=T+V;\quad T=(J\omega,\ \omega)/2,\quad V=(e,\ \gamma) \tag{1.3}$$

Let us assume that the centre of mass is not a fixed point of the motion. Then in all known cases in which system (1.1) is completely integrable on the level (1.2)—Lagrange, Kovalevskaya and Goryachev–Chaplygin—the body is dynamically symmetric, and in the last two cases the centre of mass lies in a diametral plane of the ellipsoid of inertia. We shall assume that this is the case and, moreover, that the area constant c is zero, as it is in the Goryachev–Chaplygin case.

The units of measurement and axes of inertia of the body may be chosen in such a way that $e = e_1$ is a unit basis vector, e_3 is a vector pointing along the dynamic axis of symmetry and the axial moment of inertia is unity. Then system (1.1) on the level M depends on a single non-dimensional parameter a—the ratio of the equatorial and axial moments of inertia. By the inequality between the moments of inertia, $1/2\leqslant a<\infty$. Equations (1.1) on the level (1.2) are integrable if $a = 1$ (a spherical ellipsoid of inertia), $a = 2$ (the Kovalevskaya case) and $a = 4$ (the Goryachev–Chaplygin case). It has been proved that when $c\neq 0$ and $a\geqslant 1$, Eqs (1.1) are non-integrable in Liouville's sense on the level (1.2) [3], but up to now non-integrability has not been proved in our case of $c = 0$.

Theorem 1. Let $a>4$. Then system (1.1) on the zero level M of the area integral has no analytic

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first integrals independent of the energy H or analytic symmetry groups [4]. For values of h slightly larger than the maximum 1 of the potential energy, the orbits of the system behave in a stochastic manner on the invariant subset $\{H = h\} \cap M$ of the energy level in phase space.

Proofs that Hamiltonian systems are non-integrable and possess stochastic behaviour are usually based on constructing a sufficient number of transversal homoclinic (doubly asymptotic) orbits (see e.g. [1–6]). Proofs of existence for such orbits use the Mel'nikov–Arnol'd method, which is applicable only to near-integrable systems. In the present situation this can be done only when a is close to 1, 2, 4 or ∞. We shall present a proof of the existence of homoclinic orbits based on Morse theory [7], using methods proposed in an earlier paper [8].

2. THE EXISTENCE OF HOMOCLINIC ORBITS

The potential energy V reaches its maximum value 1 at the point $P = \{\gamma = e\}$ of the Poisson sphere S^2. The corresponding equilibrium position $O = \{\omega = 0, \gamma = e\}$ of system (1.1) in the phase space $M \subseteq R^6$ is an unstable position of equilibrium with real characteristic exponents ± 1, $\pm 1/a^{1/2}$. There exist four pendulum-type orbits of system (1.1) on level M, doubly asymptotic to O, so that the rigid body rotates about a horizontal axis orthogonal to the radius-vector of the centre of mass $e = e_1$. Two pendulum-type orbits $\Gamma_{1,2} \subseteq M$, corresponding to rotation of the body about the horizontal axis e_2, are defined by

$$\gamma_2 = 0, \quad \gamma_1^2 + \gamma_3^2 = 1, \quad \omega = \omega_2 e_2, \quad \omega_2 = \pm(2(1-\gamma_1)/a)^{1/2} \tag{2.1}$$

These orbits may be derived from one another by time reversal. The two other pendulum-type orbits $\Gamma_{3,4} \subseteq M$ correspond to motion about the horizontal axis of symmetry e_3 and are defined by

$$\gamma_3 = 0, \quad \gamma_1^2 + \gamma_2^2 = 1, \quad \omega = \omega_3 e_3, \quad \omega_3 = \pm(2(1-\gamma_1))^{1/2} \tag{2.2}$$

The orbit Γ_3 corresponds to $\omega_3 > 0$ and Γ_4 to $\omega_3 < 0$.

Through the point $O \in M$ there pass two two-dimensional invariant analytic manifolds W^s and W^u—the sets of orbits of system (1.1) asymptotic to the equilibrium position O as $t \to \infty$ and $t \to -\infty$, respectively. The homoclinic orbits (HOs) $\Gamma_1 - \Gamma_4$ are the curves in which W^s and W^u intersect.

We recall that an HO is said to be transversal if the manifolds W^s and W^u intersect along it at a non-zero angle. It is true that the HOs $\Gamma_1 - \Gamma_4$ are transversal for almost all a, but this does not necessarily imply that system (1.1) is non-integrable and possesses complex behaviour on M, because the characteristic exponents of the equilibrium position O are real [9]. For example, it can be shown that when $a = 2$ all pendulum-type HOs are transversal, although the system is integrable.

By a theorem of Turayev and Shil'nikov [1], a Hamiltonian system that has an equilibrium position with real, non-vanishing characteristic exponents will display chaotic behaviour if there exist at least three transversal HOs which, as $t \to \infty$ and $t \to -\infty$, are tangent to the leading eigendirections of the equation of the first approximation, corresponding to the greatest negative characteristic exponent and least positive characteristic exponent, respectively (non-integrability was established in [5] under slightly different assumptions).

If $a > 1$ (equatorial moment of inertia greater than axial moment of inertia), the leading eigendirections corresponding to the characteristic exponents for the equilibrium position O of system (1.1) on M, which are equal to $1/a^{1/2}$ in absolute value, are the entry and exit directions of the doubly asymptotic pendulum-type orbits Γ_3 and Γ_4. The pendulum-type orbits Γ_1 and Γ_2 are tangent to the eigendirections corresponding to the characteristic exponent 1 of greatest absolute value. Therefore, all other orbits asymptotic to O, if they exist, are tangent to the leading directions at O. Thus, in order to apply the Turayev–Shil'nikov theorem it will suffice to show that Γ_3 and Γ_4 are transversal and to construct a transversal non-pendulum-type orbit doubly asymptotic to O.

Theorem 2. Let $a > 4$. Then the pendulum-type HOs Γ_3 and Γ_4 are transversal. Besides the pendulum-type orbits, there exist at least four other orbits $\Gamma_5, \ldots, \Gamma_8$ doubly asymptotic to O, where the Poisson vector for Γ_5 and Γ_6 belongs to the hemisphere $\gamma_3 > 0$, that for Γ_7 and Γ_8 to the hemisphere $\gamma_3<0$. Each of these orbits is either transversal or the contact of W^s and W^u along it is of odd order. The orbits $\Gamma_{5,6}$ and $\Gamma_{7,8}$ are obtained from one another by time reversal.

The analytic manifolds W^s and W^u on the three-dimensional energy level $\{H=1\}\cap M$ possess odd-order contact if the parts into which the intersection curves divide their own neighbourhoods in W^s lie on different sides of W^u. Using the analyticity of the system, one can show that for almost all $a>4$ the HOs we have constructed are transversal, but the variational methods used here are useless to determine the exceptional values of the parameter.

Corollary 1. For sufficiently small $h-1>0$ the intersection of a neighbourhood of the set $\cup\Gamma_i\cup\{0\}\subseteq M$, $i=3,\ldots,8$, with the energy level $\{H=h\}\cap M$ contains an invariant subset in which system (1.1) is topologically equivalent to a suspension over a topological Markov chain (or Bernoulli scheme).

When the HOs constructed are transversal, the corollary follows from Theorem 2 and [1]. The form of the transition matrix of the Markov chain was also described in [1]. It can be shown that the results of [1] carry over to the case of non-transversal HOs of odd multiplicity. That system (1.1) is non-integrable over M follows from this corollary. Non-integrability may also be proved by the methods in [5].

The remaining part of this paper is devoted to a proof of Theorem 2 via Morse theory.

3. THE MINIMALITY OF PENDULUM-TYPE ORBITS

Transforming system (1.1) on M from γ, ω variables to γ, $\gamma^{\bullet}$ variables with zero area constant, we obtain a natural system whose configuration space is the Poisson sphere $S^2\{\gamma\}$ and whose phase space is $M=TS^2$. If $(J\omega,\gamma)=0$, it follows from (1.1) that $\omega=[\gamma^{\bullet},J\gamma]/(\gamma,J\gamma)$. Using (1.3), we find the kinetic energy $T=\frac{1}{2}(J'\gamma^{\bullet},\gamma^{\bullet})/(J\gamma,\gamma)$, where J' is the adjoint matrix. According to the Maupertuis–Jacobi principle of least action, the projections of the orbit of energy $h=1$ of system (1.1) on the Poisson sphere are geodesics of the Jacobi metric

$$\|\gamma^{\bullet}\|^2=2(h-V(\gamma))T(\gamma,\gamma^{\bullet})=(1-(e,\gamma))(J'\gamma^{\bullet},\gamma^{\bullet})/(J\gamma,\gamma) \tag{3.1}$$

The Jacobi metric is degenerate at the point of maximum of potential energy $P=\{\gamma=e_1\}$. The Jacobi action of the curve $t\to\gamma(t)\in S^2$ [the length of the curve in the metric (3.1)] is

$$S(\gamma)=\int((1-\gamma_1)(\gamma_1^{\bullet 2}+\gamma_2^{\bullet 2}+a\gamma_3^{\bullet 2})/(a\gamma_1^2+a\gamma_2^2+\gamma_3^2))^{1/2}\,dt \tag{3.2}$$

Let $\Gamma\subseteq S^2$ be the equator $\{\gamma_3=0\}$ of the Poisson sphere. It is the curve described by the vector γ as it moves along the pendulum-type HOs $\Gamma_{3,4}$, and therefore a critical point of the action functional (3.2) in the class of piecewise-smooth curves with ends at P.

Remark. Since the Jacobi metric (3.1) degenerates at the point $P=\{\gamma=1\}$ of the Poisson sphere, the integrand in (3.2) is not smooth at P. The concept of a critical point of S must therefore be defined more rigorously. This will be done in Sec. 4 by modifying the domain of definition of S: instead of the set of curves with ends at P we will consider the set Ω of curves whose ends lie at a small distance (in the Jacobi metric) ε from P.

Lemma 1. If $a>4$ the equator Γ is a non-degenerate local minimum point of the Jacobi action functional in the class Ω of piecewise-smooth curves on S^2 with ends at P.

The statement of this lemma means that for all curves in Ω that lie close to Γ together with their derivatives, the Jacobi action is not less than $S(\Gamma)$—the inequality is strict for all curves near Γ except those obtained from Γ by reparametrization—and the second variation of S (in the sense of the previous remark) is non-degenerate. Lemma 1 is a corollary of the following two lemmas.

Lemma 2. Let $t\to\gamma(t)$ be a curve in Ω. Then $\partial(a^{1/2}S(\gamma))/\partial a\geq 0$, with equality only if the curve is contained in the equator Γ of the Poisson sphere.

Proof. Calculating the derivative, we see from (3.2) that

$$\partial(a^{1/2}S(\gamma))/\partial a = a^{1/2}\int (1-\gamma_1)^{1/2}\{(J\gamma,\gamma)^{-1/2}(J'\gamma^\bullet,\gamma^\bullet)^{-1/2}\gamma_3^{\bullet 2}+(J\gamma,\gamma)^{-3/2}(J'\gamma^\bullet,\gamma^\bullet)^{1/2}\gamma_3^2\}\,dt/2 \geqslant 0$$

The equality sign will hold only if $\gamma_3 = 0$. The lemma is proved.

Lemma 3. If $a = 4$ the second variation of the action functional S at a point $\Gamma \in \Omega$ is non-negative and has degree of degeneracy 1.

Proof. We will first prove the following lemma.

Lemma 4. In the Goryachev–Chaplygin case, the stable and unstable invariant manifolds W^s and W^u of the equilibrium position O are tangent to one another along the pendulum-type homoclinic orbits $\Gamma_{3,4} \subseteq M$ and the projection π: $M = TS^2 \to S^2$ of the phase space onto the configuration space maps their neighbourhoods diffeomorphically onto the Poisson sphere.

For the proof we will use the Euler–Poisson variables γ, ω. Equations (1.1) have an additional integral $F = \omega_3^2(\omega_1^2+\omega_2^2) - \omega_1\gamma_3$—the Goryachev–Chaplygin integral. On the manifolds $W^{s,u}$ all the first integrals of the problem take constant values, equal to their values at O. Consequently,

$$W^s \cup W^u \subseteq N = \{H = 1, \quad (J\omega, \gamma) = 0, \quad |\gamma| = 1, \quad F = 0\}$$

Let $L \subseteq M$ be one of the pendulum-type orbits Γ_3 or Γ_4 and $Q = \{\gamma, \omega\}$ a point of L. We will determine the tangent planes of the invariant manifolds $T_Q W^s$, $T_Q W^U \subseteq R^6$.

Let $\gamma(t)$, $\omega(t)$ be a curve in W^s or W^u with initial point $Q = (\gamma(0), \omega(0))$. Differentiating the energy integral, area integral, geometric integral and Goryachev–Chaplygin integral with respect to t at $t = 0$, we obtain

$$\omega_2\omega_2^\bullet + \gamma_1^\bullet = 0, \quad 4\gamma_1\omega_1^\bullet + 4\gamma_2\omega_2^\bullet + \omega_3\gamma_3^\bullet = 0, \quad \gamma_1\gamma_1^\bullet + \gamma_2\gamma_2^\bullet = 0 \tag{3.3}$$

and $\omega_3(\omega_1^{\bullet 2}+\omega_2^{\bullet 2}) - \omega_1^\bullet\gamma_3^\bullet = 0$. Eliminating $\gamma_3^\bullet$ and using the fact that L satisfies (2.2), we can transform the last equation to

$$((1+\gamma_1)\omega_1^\bullet + \gamma_2\omega_2^\bullet)^2 = 0 \tag{3.4}$$

Thus, the set of vectors tangent to N at Q is a two-dimensional plane in R^6 and therefore the tangent planes $T_Q W^s = T_Q W^u = T_Q N$ coincide and are given by Eqs (3.3) and (3.4).

To prove that the manifolds $W^{s,u}$ project diffeomorphically onto the Poisson sphere in the neighbourhood of Q, it will suffice to verify that for given $\gamma^\bullet \in T\gamma S^2$, Eqs (3.3) and (3.4) are uniquely solvable for $\omega^\bullet$. At Q we have $\omega_2 \neq 0$. If $\gamma_2 \neq 0$ then $\omega_1^\bullet = \gamma_3^\bullet\omega_3/4$, $\omega_2^\bullet = \gamma_3^\bullet\omega_3(1+\gamma_1)/4\gamma_2$, $\omega_3^\bullet = -\gamma_1^\bullet/\omega_3$. Since the manifolds $T_Q W^{s,u}$ depend smoothly on Q, the assertion is also true when $\gamma_2 = 0$. This proves Lemma 4.

Lemma 3 is derived from Lemma 4 as follows. Lemma 4 implies that for every point $q \in S^2$ close enough to Γ there exists a unique orbit or system (1.1), $t \to (\gamma_+(t), \gamma_+^\bullet(t)) \in TS^2 = M$, $0 \leqslant t < -\infty$, asymptotic to the point O as $t \to \infty$ and close to the HO L, such that $\gamma_+(0) = q$, and a unique orbit $t \to (\gamma_-(t), \gamma_-^\bullet(t)) \in M$, $-\infty < t \leqslant 0$, asymptotic to O as $t \to \infty$ and close to L, such that $\gamma_-(0) = q$. In addition, if $q \in \Gamma$ the function $\|\gamma_+^\bullet(0) - \gamma_-^\bullet(0)\|$ vanishes together with its derivatives. Let $S_+(q)$ be the Jacobi action of the curve γ_+ and $S_-(q)$ that of γ_-. By the formula for the variation of the action function, $\gamma_\pm^\bullet(0) = \mp \operatorname{grad} S_\pm(q)$ (the gradient is evaluated in the Riemannian metric defined by the kinetic energy on the Poisson sphere).

For any curve $\gamma \in \Omega$ sufficiently close to Γ that passes through q, we have $S(\gamma) \geqslant S(\gamma_+) + S(\gamma_-) = S_+(q) + S_-(q)$; but by what we have just proved the first and second differentials of this function vanish on Γ. This proves Lemma 3.

Corollary 2. If $a > 4$, every point $q \in S^2$ sufficiently close to Γ may be connected to P by exactly two locally minimal geodesics $\gamma_\pm$ of the Jacobi metric. These geodesics do not intersect Γ and the angle between γ_+ and γ_- in the triangle $\Gamma\gamma_+\gamma_-$ is less than π.

A geodesic is locally minimal if it is a local minimum point of the length functional S relative to all curves with the same endpoints. The existence of the geodesics $\gamma_\pm$ follows from the fact that there are no conjugate points on a locally minimal geodesic Γ. By Lemma 1, the sum $S(q)$ of lengths of the curves $\Gamma_\pm$ has a non-degenerate minimum $S(\Gamma)$ on Γ. For q near Γ, therefore, the vector $\operatorname{grad} S(q)$ points away from Γ. But it can be proved, as in the proof of Lemma 3, that this vector makes equal angles with the geodesics $\Gamma_\pm$ and is directed to the angle greater than π that they form.

Corollary 3. If $a > 4$ the invariant manifolds $W^{s,u}$ intersect transversally along $\Gamma_{3,4}$ and project diffeomorphically onto the Poisson sphere in their neighbourhood.

We have thus proved the first assertion of Theorem 2.

4. PROOF OF THE EXISTENCE OF HOMOCLINIC ORBITS

The rest of Theorem 2 will follow from a more general statement. Consider a natural Hamiltonian system with configuration space S^2, kinetic energy T, which is a positive definite quadratic form in the velocity, and potential energy V. Assume that T and V are smooth (at least C^2). Let V achieve a non-degenerate maximum h at a point $P \in S^2$. It was proved in [7] that there always exists an orbit of energy $H = T + V = h$ which is doubly asymptotic to the equilibrium position P. Its orbit $\Gamma \subseteq S^2$ is a geodesic of the Jacobi metric $\|\gamma^\bullet\|^2 = 2(h - V(\gamma))\, T(\gamma, \gamma^\bullet)$ in the domain $D = S^2 \setminus \{P\}$.

Theorem 3. Let the orbit Γ be a non-degenerate local minimum point of the Jacobi action; assume, moreover, that Γ is the reverse as $t \to \pm\infty$ of a leading asymptotic orbit. Then:

1. In each of the regions into which Γ divides S^2 there exists an orbit which is doubly asymptotic to the equilibrium position P.

2. If the system is analytic, this orbit is either transversal or of odd multiplicity.

We may assume that the sphere is oriented; we also fix an orientation of the curve Γ, so we can speak of the right and left sides of Γ. In what follows we will limit ourselves to the right region W of those into which Γ divides the sphere. The first assertion of the theorem may be sharpened as follows.

1'. Let Γ_s $(0 \leqslant s \leqslant 1)$ be the homotopy of Γ to a point, consisting of the curves on the sphere with their ends at P and lying to the right of Γ. Then there exists an orbit β, doubly asymptotic to P and lying to the right of Γ, such that

$$S(\beta) \leqslant L = \max_s S(\Gamma_s) \tag{4.1}$$

The proof will use methods proposed previously in [8]. We note that the part of the proof relating to the existence of a HO distinct from Γ may be generalized to the many-dimensional case. To prove the theorem we must show that the action functional S has critical points on Ω other than Γ. The Jacobi metric is not complete in the region D, so that the standard methods of Morse theory [7] are useless.

The following lemma was proved in [8].

Lemma 5 (the analogue of Gauss' Lemma). Let $\rho(q)$ be the distance of a point $q \in S^2$ from P in the Jacobi metric. There exists $\delta > 0$ such that ρ is a smooth function in the domain $U_\delta = \{q \in S^2: 0 < \rho(q) \leqslant \delta\}$. Every point $q \in U_\delta$ can be connected with P in U_δ by a unique geodesic γ_q of the Jacobi metric. This geodesic is of length $\rho(q)$ and it intersects all the curves $\Sigma_\varepsilon = \{p: \rho(p) = \varepsilon\}$; $0 < \varepsilon \leqslant \delta$ at right angles. In particular, $n(q) = -\gamma_q^\bullet(0)$ is the outward normal to U_δ. Every geodesic other than γ_q in U_δ is a segment of L with both ends $a, b \in \Sigma_\delta$. It intersects the curves γ_q at most at one point and is tangent to a unique curve Σ_ε, $0 < \varepsilon < \delta$. In particular, U_δ is geodesically convex, that is, its boundary Σ_δ has positive geodesic curvature.

Using the orientation of the sphere, we can define which of any two vectors making an acute angle is left and which is right. Let W be the rightmost of the domains into which Γ divides D. It will suffice to prove the existence of a homoclinic orbit in W. Let A and B be the points at which Γ intersects Σ_δ, assuming that Γ is directed from A to B.

Lemma 6. There exists a directed geodesic in W, close to Γ but not intersecting it, with its ends p and q on Σ_δ close to A and B, respectively, such that the velocity vector at p points to the left of $n(p)$ and the velocity vector at q points to the right of $-n(q)$.

Proof. Let $q \in \Sigma_\delta \cap W$ be sufficiently close to B. By Lemma 5, q can be connected to P in U_δ by a unique geodesic γ_q of length δ with initial velocity vector $-n(q)$. By Corollary 1, there is another geodesic close to Γ, connecting P to q, that does not intersect Γ and intersects Σ_δ at a point $p \in \Sigma_\delta \cap W$ near A. Its velocity vector at

p points along $n(p)$, the velocity vector at q points to the right of $-n(q)$. Consider a geodesic issuing from p with initial velocity vector pointing a little to the left of $n(p)$. By continuity, this geodesic satisfies our needs, proving the lemma.

By Lemma 5, the continuation of the geodesic γ constructed in Lemma 6 beyond the points p and q cuts Σ_δ at points p' and q' such that the segments $p'p$ and qq' are contained in U_δ, with the velocity vector at p' pointing to the right of $-n(p')$ and that at q' to the left of $n(q')$. If γ is sufficiently close to Γ, it can be shown that the segments pp' and qq' do not intersect non-leading principal asymptotic directions of the point, and hence the points A, pp', qq', B lie on Σ_δ in that order. We have used the fact that Γ is tangent to opposite leading directions as $t \to \pm\infty$.

Let K be the compact subset of W bounded by the geodesic γ and the segment of the curve $\Sigma_\delta \cap W$ between p' and q'.

Lemma 7. The boundary ∂K of K is geodesically concave. This means that any sufficiently short segment of a geodesic of the Jacobi metric with its ends on ∂K does not intersect the interior $K \backslash \partial K$ of K.

Proof. By construction, the boundary of K is the union of a segment of the geodesically concave curve Σ_δ and the geodesic γ, but both exterior angles of K are less than π.

Choose ε $(0<\varepsilon<\delta)$ so small that K does not intersect U_ε. Let Ω be the set of piecewise smooth curves β: $[0. 1] \to D$ with ends on Σ_δ. Define a functional F on Ω as follows:

$$F(\beta) = \int_0^1 \|\beta^{\cdot}(t)\|^2 dt \tag{4.2}$$

where $\| \ \|$ is the Jacobi metric. The functional F is more convenient than the action S for applications of Morse theory [7]. Its critical points that are not one-point curves are geodesics of the Jacobi metric in D, parametrized in proportion to the arc length and orthogonal to Σ_ε at their ends. By Lemma 5 they correspond to orbits doubly asymptotic to P. For Morse theory to be applicable, we still lack completeness of the Jacobi metric in D.

Let φ be a real smooth function in $(0, \infty)$ such that $\varphi(x) = 1$ for $x > 1$ and $\xi = 1/x^2$ for $0 < x < ½$. For any μ $(0<\mu<\varepsilon)$ we define a new Riemannian metric $\| \ \|_\mu$ in D:

$$\|q^{\cdot}\|_\mu = \|q^{\cdot}\| \cdot \varphi(\rho(q)/\mu) \tag{4.3}$$

This metric is identical with the Jacobi metric outside U_μ. The distance in this metric from a point $q \in U_\mu$ to Σ_μ is equal to $\mu^2/\rho(q)$, so that it is complete. The geodesics of the Jacobi metric that connect points of U_δ with P are geodesics of the metric (4.3). Let F_μ be the functional defined on Ω by formula (4.2) with the Jacobi metric replaced by (4.3).

Lemma 8. For any $0<\mu<\varepsilon$, F_μ has a critical point $\beta \in \Omega$ such that β is a curve in W, which intersects K and

$$F_\mu(\beta) \leqslant C = (L-2\varepsilon)^2 \tag{4.4}$$

The proof uses standard methods of Morse theory and follows the same procedure as in [8]; the details will therefore be omitted. The family of curves Γ_s with ends at P defines a family $\beta_s \in \Omega$, $0 \leqslant s \leqslant 1$, which shrinks the segment $\beta_0 = AB$ of Γ to the point β_1 and is such that for sufficiently small μ

$$\max_s F_\mu(\beta_s) = \max_s F(\beta_s) \leqslant C \tag{4.5}$$

Choose a sufficiently fine partition of the interval [0, 1] and let X be the set of polygonal geodesics β of the metric (4.3) in Ω that correspond to this partitition, such that $F_\mu(\beta) \leqslant C$ [7]. Then X is a smooth compact manifold with boundary $\{f = C\}$, where $f = F_\mu$ is a smooth function on X whose critical points are geodesics of the metric (4.3) orthogonal to Σ_ε at their ends. Let Y be the set of curves in X that lie in $W \cup \Gamma$ and Z the set of curves in Y that intersect K. Then the assertion of the lemma means that f has a critical point on Z.

Let $g_t: X \to X$, $t \geqslant 0$, denote the transformation semigroup generated by the vector field $-\text{grad} f$ (a Reimannian metric on X is defined in [7]). Since the set $W \backslash K$ is geodesically convex in the metric (4.3), by Lemma 6, it follows that Y and $Y \backslash Z$ are invariant with respect to the semigroup g_t [8]. Approximating the curves of the family β_s by polygonal geodesics, we may assume that $\beta_s \in Y$, and (4.5) is true. If f has no critical points on $Y \cap Z$, then by compactness $\|\text{grad} f\| \geqslant a > 0$ on that set. Then during a time $T = C/a$ the set $g_T(Y)$ will not intersect Z (the invariance of $Y\setminus Z$ is essential here). Since β_0 is a geodesic, $g_T(\beta_0) = \beta_0$. We obtain a homotopy $g_T(\beta_0)$ of β_0 to a point, consisting of curves in the hemisphere W that do not intersect K. This is impossible.

Proof of Theorem 3. Let β be the geodesic of the metric (4.3) constructed in Lemma 7 and let $\beta(\tau)\in K$, $0<\tau<1$. We may assume that β is parametrized in proportion to the arc length. Continue β to the left and right of τ, up to the nearest point of intersection with Σ_μ. We obtain a geodesic γ_μ of the Jacobi metric with ends on Σ_μ; moreover, by construction, $S(\gamma_\mu)\leqslant L$ and γ_μ intersects K. We have here used the fact that β is orthogonal to Σ_ε at its ends, so that if, say, $\beta([0,\tau])$ does not intersect Σ_μ, then the continuation of $\beta(t)$ into the domain of negative t gives a curve β: $[-(\varepsilon-\mu)/\|\beta^\bullet\|, 0]$ that connects Σ_μ with Σ_ε.

We may assume that $t\rightarrow\gamma_\mu(t)$ is parametrized by arc length and $\gamma_\mu(0)\in K$. Let $\mu\rightarrow 0$. We can extract subsequences from $\gamma_\mu(0)$, $\gamma_\mu{}^\bullet(0)$ that converge respectively to a point in K and to a unit vector. The geodesic of the Jacobi metric with the appropriate initial condition corresponds to an orbit that is doubly asymptotic to P [8]. This proves the first part of Theorem 3.

Assume now that the system is analytic. Then the invariant manifolds $W^{s,u}$ of the equilibrium position are analytic submanifolds of the phase space. Since Γ is a transversal doubly asymptotic orbit, they do not coincide, and consequently their curves of intersection have finite multiplicites and are isolated. The Jacobi action S corresponding to an asymptotic orbit is defined on each of these manifolds and the sets $\{S\leqslant C\}\cap W^{s,u}$ are compact. Thus the number of orbits of the Jacobi action $S\leqslant C$ doubly asymptotic to P is finite.

Lemma 9. Suppose that all orbits homoclinic to P in the domain W in which the action is not greater than C are not transversal and have even multiplicity. Then there exists a smooth function V', as close to V as desired and equal to V outside $W\setminus U_\varepsilon$, such that the perturbed system with potential energy V' and the same kinetic energy as before has no orbits of action which are not greater than C homoclinic to P and W.

Let us first see how to deduce the second part of Theorem 3 from this lemma. Suppose that all HOs in W of action not greater than L have even multiplicity [L is defined by (4.1)]. Then this is true if L is replaced by a C slightly greater than L. By Lemma 9 the perturbed system has no HOs in W of action at most C. By the first part of Theorem 3, if V' is close enough to V the system with potential energy V' will have an orbit, distinct from Γ and doubly asymptotic to P, with action at most C. This contradiction proves the theorem.

We shall only sketch the proof of Lemma 9. For any homoclinic orbit in W of even multiplicity, we have a geodesic γ of the Jacobi metric in W, orthogonal to Σ_ε at its ends p and q and intersecting K. We may assume that q is not a focal point for p, i.e. the manifold $W^{s,u}$ projects diffeomorphically onto D in the neighbourhood of q. Then, for all geodesics issuing from a neighbourhood U of p on Σ_ε along the normal n to Σ_ε, the tangent vector at a point of intersection with Σ_ε near q points to one side of the normal $-n$, say to the right. An exception is the tangent vector to γ at q, which points along $-n(q)$.

Let Q be a point on γ near q. Replace V by $V' = V+\alpha f$, where the smooth function f does not vanish in a small neighbourhood of Q and the vector $\operatorname{grad} f(Q)$ is orthogonal to γ and points to the right. Then it can be shown that, if $\alpha>0$ is small enough, the velocity vectors of all the above geodesics of the Jacobi metric that begin in U at a point of intersection with Σ_ε near q point to the right of the normal $-n$. Hence it follows that in a sufficiently small neighbourhood of γ, independent of α, there are no doubly asymptotic orbits of the perturbed system of action not greater than C. Repeating the construction for each of the finite number of HOs, we get a contradiction, since HOs of the perturbed system of action bounded by a constant C may exist only in an arbitrarily small neighbourhood of the unperturbed HOs.

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OPTIMAL CONTROL OF THE ROTATION OF A SOLID WITH A FLEXIBLE ROD†

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Two optimal control problems may arise when a solid with a rigidly attached rod is rotating in a plane: how to steer the system from an initial phase state to a terminal state so as to minimize a quadratic cost functional, and time-optimal control. A new method is proposed for constructing optimal controls, based on the results of [1, 2] and methods of functional analysis. The controls are constructed as series in terms of a certain system of functions. Using the Voigt model of matter, some consideration is also given to a system with a viscoelastic rod and analogous results are obtained. The method is applicable to the problem of steering the system from an initial to a terminal phase state so as to minimize any convex functional of the control.

1. STATEMENT OF THE PROBLEM

WE WILL study a mechanical system consisting of a solid with a rigidly attached elastic rod of constant cross-section and mass uniformly distributed along its length. At the centre of mass of the solid we place an inertial system of coordinates $OX_1Y_1Z_1$, oriented so that the central axis of the rod lies in the $O_1X_1Y_1$ plane. The system may rotate about the O_1Z_1 axis, about which the torque $M'(t')$ of the controlling forces is applied. Attached to the solid is a system of coordinates $O'X'Y'Z'$, with its origin at the point of insertion of the rod, with the $O'X'$ axis pointing along the tangent to the neutral axis of the rod at the point of insertion and the $O'Z'$ axis parallel to the O_1Z_1 axis. The position of the entire system is uniquely described by the angle of deflection $\theta(t')$ (between the $O'X'$ and O_1X_1 axes) and the amount $y'(x', t')$ of transverse deformation of the rod at a point x' and time t' (Fig. 1).

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